MATB44 Midterm Notes

1. Introduction:

- The different forms/types of differential eqns we've seen are:

a) Linear First Order Differential Eqns
b) Separable Eqns
c) Exact Eqns
d) Homogeneous Eqns with Constant Coefficients
e) Reduction of Order
f) Euler Equation
g) Non-homogeneous Equations

2. Linear First Order Differential Eqns:

- We usually write the general first order linear differential eqn in the form

$$\frac{dy}{dt} + p(t)y = g(t) \quad (1)$$

where p and g are given functions of the independent variable t.

- Another way we can the eqn is

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \quad (2)$$

where P, Q and G are given functions.

If $P(t) \neq 0$, we can (2) to (1) by dividing both sides of (2) by $P(t)$.

- Note: First order means we have up to y'. I.e. We don't have y'' or y''', etc.

- Note: $P(t)$, $Q(t)$, $G(t)$, $p(t)$, $g(t)$ are known while y is unknown.

- If we are given an eqn in the form of (2), we can sometimes assume that

$$P(t)\frac{dy}{dt} + Q(t)y = \frac{d}{dt}(f(t)y)$$

for some function f we know.

Then:

a) $(f(t)y)' = G(t)$

b) $f(t)y = \int G(t)\,dt$

$$y = \frac{\int G(t)\,dt}{f(t)}$$

Note: This method only works sometimes.

E.g. 1 Solve $t^2y' + 2ty = t^3$

Soln:

Here, $t^2y' + 2ty = \underbrace{(t^2y)'}_{\uparrow \text{ Our } (f(t)y)'}$

$(t^2y)' = t^3$

$t^2y = \int t^3\,dt$

$= \frac{t^4}{4} + c$

$y = \frac{1}{t^2}\left(\frac{t^4}{4} + c\right)$

- Solns to first order diff eqns always have 1 numerical parameter which varies and is sometimes arbitrary. I.e. There's 1 constant.
Solns to second order diff eqns always have 2 constants. The constants govern the initial condition.

- Another way to solve first order linear diff eqns is to use an integrating factor.
Note: If you're using this method, the coefficient of y' must be 1.
I.e. The form of the eqn must be
$y' + p(t)y = g(t)$.

E.g. 2 Solve $t^2y' + 2ty = t^3$

Soln:
1. Divide both sides of the eqn by t^2.

$$y' + \frac{2y}{t} = t \quad (1)$$

2. Multiply both sides of the eqn by $\mu(t)$ the integrating factor.

$$\mu y' + \frac{2\mu y}{t} = \mu t \quad (2)$$

3. Equate the LHS of (2) with $(\mu y)'$ and solve for μ.

$$\cancel{\mu y'} + \frac{2\mu y}{t} = (\mu y)'$$

Note: $\mu y'$ always cancels out.

$$= \mu' y + \cancel{\mu y'}$$

$$\frac{2\mu \cancel{y}}{t} = \mu' \cancel{y}$$

Note: y always cancels out.

$$\frac{2}{t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

Note: $(\ln(f(x)))' = \frac{f'(x)}{f(x)}$

Hence, $\frac{\mu'}{\mu} = (\ln(\mu))'$

$$\int \frac{2}{t}\, dt = \int (\ln(\mu))'\, dt$$

$$2\ln|t| + C_1 = \ln(\mu) + C_2$$

$$2\ln|t| + C_1 - C_2 = \ln(\mu)$$

$$2\ln|t| + C = \ln(\mu)$$

$$e^{2\ln|t| + C} = e^{\ln(\mu)}$$

Note: $x = a^{\log_a x}$

Since $\ln x = \log_e x$, $e^{\ln x} = x$.

$$\mu = e^{2\ln|t|} \cdot \underbrace{e^C}_{C'}$$

Note: $e^{a+b} = e^a \cdot e^b$ or $a^{m+n} = a^m \cdot a^n$

$$= C'(e^{\ln|t|})^2$$

Note: $a^{n \cdot m} = (a^m)^n$

$$= C' \cdot t^2$$

Let $C' = 1$

$$\mu = t^2 \quad (3)$$

4. After we solve for $\mu(t)$, go back to (2) and do $(\mu y)' = \text{RHS}$ and solve for y.

$$(\mu y)' = \mu t$$
$$(t^2 y)' = t^3$$
$$\int (t^2 y)' dt = \int t^3 \, dt$$
$$t^2 y + c_1' = \frac{t^4}{4} + c_2'$$
$$t^2 y = \frac{t^4}{4} + c_2' - c_1'$$
$$= \frac{t^4}{4} + C$$
$$y = \frac{1}{t^2}\left(\frac{t^4}{4} + C\right)$$

E.g. 3 Solve $y' + \frac{y}{2} = \frac{e^{t/3}}{2}$

Soln:

1. Since the coefficient of y' is already 1, we can skip this step.

2. $\mu y' + \frac{\mu y}{2} = \frac{\mu e^{t/3}}{2}$

3. $\cancel{\mu y'} + \frac{\mu y}{2} = (\mu y)'$

$$= \mu' y + \cancel{\mu y'}$$
$$\frac{\mu \cancel{y}}{2} = \mu' \cancel{y}$$
$$\frac{1}{2} = \frac{\mu'}{\mu}$$

$$\frac{1}{2} = (\ln(\mu))'$$

$$\int \frac{1}{2}\,dt = \int (\ln(\mu))'\,dt$$

$$\frac{t}{2} + c_1 = \ln(\mu) + c_2$$

$$\frac{t}{2} + c_1 - c_2 = \ln(\mu)$$

$$\frac{t}{2} + c = \ln(\mu)$$

$$\mu = e^{\frac{t}{2}+c}$$
$$= e^{t/2} \cdot e^{c}$$
$$= c' \cdot e^{t/2}$$

Let $c' = 1$

$$\mu = e^{t/2}$$

4. $(\mu y)' = \dfrac{\mu e^{t/3}}{2}$

$$(e^{t/2} y)' = \frac{e^{t/2} \cdot e^{t/3}}{2}$$
$$= \frac{e^{5t/6}}{2}$$

$$\int (e^{t/2} y)'\,dt = \int \frac{e^{5t/6}}{2}\,dt$$

To solve $\int \frac{e^{5t/6}}{2}\,dt$, I'll use u-sub.

Let $u = \frac{5t}{6}$

$\frac{du}{dt} = \frac{5}{6}$

$dt = \frac{6}{5} du$

$\int \frac{6}{10} e^{u} \, du$

$= \frac{3}{5} e^{u} + C_2$

$= \frac{3}{5} e^{5t/6} + C_2$

$e^{t/2} y + C_1 = \frac{3}{5} e^{5t/6} + C_2$

$e^{t/2} y = \frac{3}{5} e^{5t/6} + C_2 - C_1$

$= \frac{3}{5} e^{5t/6} + C$

$y = \frac{1}{e^{t/2}} \left(\frac{3}{5} e^{5t/6} + C \right)$

E.g. 4 Solve $ty' + 2y = 4t^2$, $y(1) = 2$

Soln:

$y' + \frac{2y}{t} = 4t$

$\mu y' + \frac{2\mu y}{t} = 4\mu t$

$$\cancel{\mu y'} + \frac{2\mu y}{t} = (\mu y)'$$

$$= \mu' y + \cancel{\mu y'}$$

$$\frac{2\mu \cancel{y}}{t} = \mu' \cancel{y}$$

$$\frac{2}{t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{2}{t}\, dt = \int (\ln(\mu))'\, dt$$

$$2\ln|t| + c_1 = \ln(\mu) + c_2$$

$$2\ln|t| + c_1 - c_2 = \ln(\mu)$$

$$2\ln|t| + c = \ln(\mu)$$

$$\mu = e^{2\ln|t| + c}$$

$$= e^{c}(e^{\ln|t|})^2$$

$$= c' t^2$$

let $c' = 1$

$$\mu = t^2$$

$$(\mu y)' = 4\mu t$$

$$(t^2 y)' = 4t^3$$

$$\int (t^2 y)'\, dt = \int 4t^3\, dt$$

$$t^2 y + c_1 = t^4 + c_2$$

$$t^2 y = t^4 + c$$

$$y = t^2 + \frac{c}{t^2}$$

$$y(1) = 2$$

$$2 = 1^2 + \frac{c}{1^2}$$

$$c = 1$$

$$y = t^2 + \frac{1}{t^2}$$

E.g. 5 Solve $(1+t^2)y' + 4ty = (1+t^2)^{-2}$

Soln:

$$y' + \frac{4ty}{1+t^2} = (1+t^2)^{-3}$$

$$\mu y' + \frac{4\mu t y}{1+t^2} = \mu(1+t^2)^{-3}$$

$$\mu y' + \frac{4\mu t y}{1+t^2} = (\mu y)'$$

$$= \mu' y + \mu y'$$

$$\frac{4t}{1+t^2} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{4t}{1+t^2}\,dt = \int (\ln(\mu))'\,dt$$

To solve $\int \frac{4t}{1+t^2}\,dt$, I'll use u-sub.

Let $u = 1+t^2$

$\frac{du}{dt} = 2t$

$dt = \frac{du}{2t}$

$$\int \frac{2}{u}\,du = 2\ln|u| + C_1$$

$$= 2\ln|1+t^2| + C_1$$

$2\ln|1+t^2| + C_1 = \ln(\mu) + C_2$

$2\ln|1+t^2| + C = \ln(\mu)$

$$\mu = e^{2\ln|1+t^2|+c}$$
$$= e^c (e^{\ln|1+t^2|})^2$$
$$= c'(1+t^2)^2$$

Let $c'=1$

$$\mu = (1+t^2)^2$$

$$(\mu y)' = \mu (1+t^2)^{-3}$$
$$= (1+t^2)^{-1}$$
$$\int (\mu y)'\,dt = \int \frac{1}{1+t^2}\,dt$$

$$(1+t^2)^2 y + c_1 = \arctan(t) + c_2$$
$$(1+t^2)^2 y = \arctan(t) + c$$
$$y = \frac{\arctan(t)+c}{(1+t^2)^2}$$

E.g. 6 Solve $ty' + (t+1)y = t$, $y(\log 2) = 1$

Soln:

$$y' + \frac{(t+1)y}{t} = 1$$

$$\mu y' + \frac{\mu(t+1)y}{t} = \mu$$

$$\cancel{\mu y'} + \frac{\mu(t+1)\cancel{y}}{t} = (\mu y)'$$
$$= \mu'\cancel{y} + \cancel{\mu y'}$$
$$\frac{t+1}{t} = \frac{\mu'}{\mu}$$
$$= (\ln(\mu))'$$

$$\int \frac{t+1}{t}\, dt = \int (\ln(\mu))'\, dt$$

$$\int 1\, dt + \int \tfrac{1}{t}\, dt = \ln(\mu) + C_2$$

$$t + \ln|t| + C_1 = \ln(\mu) + C_2$$

$$t + \ln|t| + C = \ln(\mu)$$

$$\mu = e^{t + \ln|t| + c}$$

$$= e^c \cdot e^t \cdot e^{\ln|t|}$$

$$= c' \cdot e^t \cdot t$$

let $c' = 1$

$$\mu = te^t$$

$$(\mu y)' = \mu$$

$$(te^t y)' = te^t$$

$$\int (te^t y)'\, dt = \int te^t\, dt$$

To solve $\int te^t\, dt$, I will do integration by parts.
Let $u = t$, $v = e^t$

$$u \int v\, dt - \int (u' \int v\, dt)\, dt$$

$$t \int e^t\, dt - \int (t' \int e^t\, dt)\, dt$$

$$te^t - e^t + C_2$$

$$te^t y + C_1 = te^t - e^t + C_2$$

$$te^t y = te^t - e^t + C$$

$$y = 1 - \frac{1}{t} + \frac{c}{te^t}$$

$y(\log 2) = 1$

$$1 = 1 - \frac{1}{\log 2} + \frac{c}{(\log 2)(2)}$$

$$0 = \frac{c}{2\log 2} - \frac{1}{\log 2}$$

$$\frac{1}{\log 2} = \frac{c}{2\log 2}$$

$$C = 2$$

$$y = 1 - \frac{1}{t} + \frac{2}{te^t}$$

E.g. 7 Solve $ty' + y = t\cos t^2$

Soln:

$$y' + \frac{y}{t} = \cos t^2$$

$$\mu y' + \frac{\mu y}{t} = \mu \cos t^2$$

$$\cancel{\mu y'} + \frac{\mu \cancel{y}}{t} = (\mu y)'$$

$$= \mu' \cancel{y} + \cancel{\mu y'}$$

$$\frac{1}{t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{1}{t}\, dt = \int (\ln(\mu))'\, dt$$

$$\ln|t| + c_1 = \ln(\mu) + c_2$$

$$\ln|t| + c = \ln(\mu)$$

$$\mu = e^{\ln|t| + c}$$

$$= e^c \cdot e^{\ln|t|}$$

$$= c' \cdot t$$

let $c' = 1$

$$\mu = t$$

$(\mu y)' = \mu \cos t^2$

$(ty)' = t\cos t^2$

$\int (ty)' dt = \int t \cos t^2 \, dt$

To solve $\int t \cos t^2 \, dt$, I'll use integration by sub.

Let $u = t^2$

$\frac{du}{dt} = 2t$

$dt = \frac{du}{2t}$

$$\frac{\int \cos(u)\, du}{2} = \frac{\sin(u)}{2} + c_2$$

$$= \frac{\sin(t^2)}{2} + c_2$$

$$ty + c_1 = \frac{\sin(t^2)}{2} + c_2$$

$$ty = \frac{\sin(t^2)}{2} + c$$

$$y = \frac{\sin(t^2)}{2t} + \frac{c}{t}$$

E.g. 8 Solve $(1+t^2)y' + 4ty = \frac{2}{1+t^2}$, $y(0)=1$

Soln:

$$y' + \frac{4ty}{1+t^2} = \frac{2}{(1+t^2)^2}$$

$$\mu y' + \frac{4\mu t y}{1+t^2} = \frac{2\mu}{(1+t^2)^2}$$

$$\cancel{\mu y'} + \frac{4\mu t \cancel{y}}{1+t^2} = (\mu y)'$$

$$= \mu' \cancel{y} + \cancel{\mu y'}$$

$$\frac{4t}{1+t^2} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int \frac{4t}{1+t^2}\, dt = \int (\ln(\mu))'\, dt$$

To solve $\int \frac{4t}{1+t^2}\, dt$, I'll use u-sub.

Let $u = 1+t^2$

$$\frac{du}{dt} = 2t$$

$$dt = \frac{du}{2t}$$

$$\int \frac{2}{u}\, du = 2\ln|u| + C_1$$

$$2\ln|u| + C_1 = \ln(\mu) + C_2$$

$$\ln(\mu) = 2\ln|1+t^2| + C$$

$$\mu = (1+t^2)^2$$

$$(\mu y)' = \frac{2\mu}{(1+t^2)^2}$$

$$((1+t^2)^2 y)' = 2$$

$$\int ((1+t^2)^2 y)' \, dt = \int 2 \, dt$$

$$(1+t^2)^2 y + c_1 = 2t + c_2$$

$$(1+t^2)^2 y = 2t + c$$

$$y = \frac{2t + c}{(1+t^2)^2}$$

$$1 = \frac{c}{1} \rightarrow c = 1$$

$$y = \frac{2t+1}{(1+t^2)^2}$$

E.g. 9 Solve $y' - y = 2te^{2t}$, $y(0) = 1$

Soln:

$$\mu y' - \mu y = \mu 2te^{2t}$$

$$\cancel{\mu y'} - \mu y = (\mu y)'$$

$$= \mu' y + \cancel{\mu y'}$$

$$-1 = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int -1 \, dt = \int (\ln(\mu))' \, dt$$

$$-t + c_1 = \ln(\mu) + c_2$$

$$-t + c = \ln(\mu)$$

$$\mu = e^{-t+c}$$

$$= e^c \cdot e^{-t}$$

$$= c' \cdot e^{-t}$$

$$= e^{-t}$$

$$(\mu y)' = 2te^t$$

$$\int (\mu y)' \, dt = \int 2te^t \, dt$$

$$= 2 \int te^t \, dt$$

$$\mu y + c_1 = 2(te^t - e^t) + c_2$$

$$\mu y = 2(te^t - e^t) + c$$

$$y = \frac{2(te^t - e^t) + c}{e^{-t}}$$

$y(0) = 1$

$$1 = 2(e^0)(0e^0 - e^0) + ce^0$$

$$= -2 + c$$

$$c = 3$$

$$y = 2e^t(te^t - e^t) + 3e^t$$

3. Separable Eqns:

- A separable diff eqn is any diff eqn that can be written as:

a) $M(x) + N(y)\dfrac{dy}{dx} = 0$

OR

b) $M(x)\,dx + N(y)\,dy = 0$

- To solve separable eqns, we first move all the Y's to one side of the eqn and all the X's to the other side.

I.e. $N(y)\,dy = M(x)\,dx$

Next, we integrate both sides.

I.e. $\int N(y)\,dy = \int M(x)\,dx$

E.g. 10 Solve $y' = \dfrac{x^2}{y(1+x^3)}$

Soln:

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$$

$$y\,dy = \frac{x^2}{1+x^3}\,dx$$

$$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx$$

To solve $\int \frac{x^2}{1+x^3}\,dx$, I'll do u-sub.

Let $u = 1+x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$\int \frac{1}{3u}\,du = \frac{1}{3}\ln|u| + c_2$

$= \frac{1}{3}\ln|1+x^3| + c_2$

$\frac{y^2}{2} + c_1 = \frac{1}{3}\ln|1+x^3| + c_2$

$\frac{y^2}{2} - \frac{\ln|1+x^3|}{3} = c$

E.g. 11 Solve $y' + y^2\sin(x) = 0$

Soln:

$\frac{dy}{dx} + y^2\sin(x) = 0$

$\frac{dy}{dx} = -y^2\sin(x)$

$\frac{1}{y^2}\,dy = -\sin(x)\,dx$

$\int \frac{1}{y^2}\,dy = \int -\sin(x)\,dx$

$\frac{-1}{y} + c_1 = \cos(x) + c_2$

$\frac{-1}{y} - \cos(x) = c$

E.g. 12 Solve $y' = \frac{2x}{1+2y}$, $y(2)=0$

Soln:

$$\frac{dy}{dx} = \frac{2x}{1+2y}$$

$$1+2y\,dy = 2x\,dx$$

$$\int 1+2y\,dy = \int 2x\,dx$$

$$y+y^2+C_1 = x^2+C_2$$

$$y+y^2-x^2 = C$$

$$0+0^2-2^2 = C$$

$$C = -4$$

$$y+y^2-x^2 = -4$$

E.g. 13 Solve $y' = 2y^2 + xy^2$, $y(0)=1$

Soln:

$$\frac{dy}{dx} = y^2(2+x)$$

$$\frac{1}{y^2}\,dy = 2+x\,dx$$

$$\int \frac{1}{y^2}\,dy = \int 2+x\,dx$$

$$\frac{-1}{y} + C_1 = 2x + \frac{x^2}{2} + C_2$$

$$\frac{-1}{y} - 2x - \frac{x^2}{2} = C$$

$$-1 = C$$

$$\frac{-1}{y} - 2x - \frac{x^2}{2} = -1$$

E.g. 14 Solve $x^2y^2y' + 1 = y$

Soln:

$x^2y^2\dfrac{dy}{dx} = y-1$

$x^2\dfrac{dy}{dx} = \dfrac{y-1}{y^2}$

$x^2\,dy = \dfrac{y-1}{y^2}\,dx$

$\dfrac{y^2}{y-1}\,dy = \dfrac{1}{x^2}\,dx$

$\displaystyle\int\frac{y^2}{y-1}\,dy = \int\frac{1}{x^2}\,dx$

To solve $\displaystyle\int\frac{y^2}{y-1}\,dy$, I'll use u-sub.

Let $u = y-1 \longrightarrow y = u+1$

$\dfrac{du}{dy} = 1$

$dy = du$

$\displaystyle\int\frac{(u+1)^2}{u}\,du$

$\displaystyle= \int\frac{u^2+2u+1}{u}\,du$

$\displaystyle= \int\frac{u^2}{u}\,du + \int\frac{2u}{u}\,du + \int\frac{1}{u}\,du$

$\displaystyle= \int u\,du + \int 2\,du + \int\tfrac{1}{u}\,du$

$= \dfrac{u^2}{2} + 2u + \ln|u| + C_1$

$$\frac{(y-1)^2}{2} + 2(y-1) + \ln|y-1| + C_1 = \frac{-1}{x} + C_2$$

$$\frac{(y-1)^2}{2} + 2(y-1) + \ln|y-1| + \frac{1}{x} = C$$

E.g. 15 Solve $y^2 + x^2y' = xyy'$

Soln:

$$y^2 = xyy' - x^2y'$$

$$= y'(xy - x^2)$$

$$1 = \frac{y'(xy-x^2)}{y^2}$$

$$\frac{1}{y'} = \frac{xy - x^2}{y^2}$$

$$= \left(\frac{x}{y}\right) - \left(\frac{x}{y}\right)^2 \leftarrow \text{This is a homogeneous eqn.}$$

Let $v = \frac{y}{x}$. This means that $y = vx$ and $y' = (vx)'$.

$$\frac{1}{(vx)'} = \frac{1}{v} - \frac{1}{v^2}$$

$$= \frac{v-1}{v^2}$$

$$\frac{v^2}{v-1} = (vx)'$$

$$= v'x + v$$

$$\frac{v^2}{v-1} - v = x\frac{dv}{dx}$$

$$\frac{v^2 - v(v-1)}{v-1} = x\frac{dv}{dx}$$

$$\frac{v}{v-1} = x\frac{dv}{dx}$$

$$\frac{v}{v-1}\,dx = x\,dv$$

$$\frac{1}{x}\,dx = \frac{v-1}{v}\,dv$$

$$\int \frac{1}{x}\,dx = \int \frac{v-1}{v}\,dv$$

$$\ln|x| + C_1 = v - \ln|v| + C_2$$

$$C = v - \ln|v| - \ln|x|$$

$$= \frac{y}{x} - \ln\left|\frac{y}{x}\right| - \ln|x|$$

Note: This is a homogeneous equation. A first order diff eqn is homogeneous if it can be written in this form: $y' = f(y/x)$. To make it a separable eqn, we introduce a new var $v = \frac{y}{x}$. Since $v = \frac{y}{x}$, $y = vx$ and $y' = (vx)'$

$$= v'x + vx'$$

$$= v'x + v$$

At the end, we must sub $\frac{y}{x}$ for v.

E.g. 16 Solve $y' = \frac{x^2 + xy + y^2}{x^2}$

Soln:

$y' = 1 + (\frac{y}{x}) + (\frac{y}{x})^2 \leftarrow$ Homogeneous Eqn

Let $v = \frac{y}{x}$, $y = vx$ and $y' = v'x + v$

$$v'x + v = 1 + v + v^2$$

$$v'x = 1 + v^2$$

$$\frac{dv}{dx}\,x = 1 + v^2$$

$$\frac{1}{1+v^2}\,dv = \frac{1}{x}\,dx$$

$$\arctan(v) + C_1 = \ln|x| + C_2$$

$$\arctan\left(\frac{y}{x}\right) - \ln|x| = C$$

4. Exact Eqns:

- Suppose we have the following eqn
$M(x,y) + N(x,y)y' = 0$.

- If there's a function $F(x,y)$ s.t.
a) $F_x = M(x,y)$ and
b) $F_y = N(x,y)$
then, we say $F(x,y)$ is an exact eqn.

- Recall that $dF = F_x\,dx + F_y\,dy$
Furthermore, since F is an exact eqn, then we know that:
a) $M(x,y) = F_x$
b) $N(x,y) = F_y$
Replacing $M(x,y)$ with F_x and $N(x,y)$ with F_y from the first eqn, we get $F_x + F_y y' = 0$

$$\rightarrow F_x + F_y \frac{dy}{dx} = 0$$

$$\rightarrow \underbrace{F_x\,dx + F_y\,dy}_{dF} = 0$$

Hence, $dF = 0$. This implies that $F(x,y) = C$ for some constant C.

- We have a test to determine whether or not F is an exact eqn. If F is an exact eqn, then $M_y = N_x$.

Proof:
Suppose that F is an exact eqn. Then,
a) $M = F_x$
b) $N = F_y$
c) $F_{xy} = F_{yx}$ (Mixed derivative rule)

From c), $F_{xy} = (F_x)_y$ and $F_{yx} = (F_y)_x$
$= M_y$ $= N_x$

Since $F_{xy} = F_{yx}$, we know that $M_y = N_x$.

Note: Sometimes, $M_y \neq N_x$. In this case, we'll need to use an integrating factor.

- Steps for solving an exact eqn:

1. See if $M_y = N_x$
   a) If yes
      i) Let $M = F_x$ and $N = F_y$.
      ii) From $M = F_x$, $F = \int M\,dx$.
      iii) We know that $F_y = N$.
      However, we also know that $F_y = \partial_y F$.
      Do $\partial_y F = N$
      Note: After simplifying $\partial_y F = N$, all terms with x should be gone.
      iv) Solve for F

   b) If No
      i) Multiply both sides of the eqn by μ.
      ii) Do $\partial_y(\mu M) = \partial_x(\mu N)$ and solve for μ.
      iii) The new function should be exact now. Follow the steps in part a to solve it.

E.g. 17 Solve $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$

Soln:

$(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0$

$M = y\cos x + 2xe^y$

$N = \sin x + x^2e^y - 1$

Check if $M_y = N_x$.

$M_y = \partial_y(M)$
$= \partial_y(y\cos x + 2xe^y)$
$= \cos x + 2xe^y$

$N_x = \partial_x(N)$
$= \partial_x(\sin x + x^2e^y - 1)$
$= \cos x + 2xe^y$

$M_y = N_x$

Let $M = \partial_x F$, and $N = \partial_y F$.

$\partial_x F = y\cos x + 2xe^y$
$F = \int y\cos x + 2xe^y \, dx$
$= y\sin x + e^y x^2 + \underline{c(y)}$

Note: C might be a function of y bc when we differentiate w.r.t. x, y is treated as a constant.

$N = \partial_y F$

$\sin x + x^2e^y - 1 = \partial_y(\overbrace{y\sin x + e^y x^2 + c(y)})$
$= \sin x + x^2e^y + c'(y)$
$-1 = c'(y)$

Note: At this point, all terms with x should be gone.

$C = \int -1 \, dy$
$= -y + C_1$
Let $C_1 = 0$

$F = y\sin x + e^y x^2 - y = C$ ← Final Soln

E.g. 18 Solve $\left(\frac{y}{x}+6x\right)+(\log x-2)y'=0$

Soln:

$$\left(\frac{y}{x}+6x\right)dx+\left(\log x-2\right)dy=0$$

$$M=\frac{y}{x}+6x$$

$$N=\log x-2$$

Check if $M_y=N_x$

$$M_y=\partial_y M$$
$$=\partial_y\left(\frac{y}{x}+6x\right)$$
$$=\frac{1}{x}$$

$$N_x=\partial_x N$$
$$=\partial_x(\log x-2)$$
$$=\frac{1}{x}$$

$$M_y=N_x$$

Let $M=\partial_x F$, $N=\partial_y F$.

$$\partial_x F=\frac{y}{x}+6x$$
$$F=\int\frac{y}{x}+6x\,dx$$
$$=y\ln|x|+3x^2+C(y)$$

$$N=\partial_y F$$
$$\log x-2=\partial_y(y\ln|x|+3x^2+C(y))$$
$$=\ln|x|+C'(y)$$
$$C'(y)=-2$$
$$C=\int-2\,dy$$
$$=-2y$$

$$F=y\ln|x|+3x^2-2y=C$$

E.g. 19 Solve $(2xy - 9x^2) + (2y + x^2 + 1)y' = 0$

Soln:

$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$

$M = 2xy - 9x^2 \qquad N = 2y + x^2 + 1$

Check if $M_y = N_x$

$M_y = \partial_y M$
$\quad = \partial_y (2xy - 9x^2)$
$\quad = 2x$

$N_x = \partial_x N$
$\quad = \partial_x (2y + x^2 + 1)$
$\quad = 2x$

$M_y = N_x$

Let $M = \partial_x F$, $N = \partial_y F$

$\partial_x F = M$
$\quad = 2xy - 9x^2$
$F = \int 2xy - 9x^2\, dx$
$\quad = x^2 y - 3x^3 + c(y)$

$N = \partial_y F$
$2y + x^2 + 1 = \partial_y (x^2 y - 3x^3 + c(y))$
$\quad = x^2 + c'(y)$
$c'(y) = 2y + 1$
$c = \int 2y + 1\, dy$
$\quad = y^2 + y + c_1$

$F = x^2 y - 3x^3 + y^2 + y = C$

E.g. 20 Solve $(3xy+y^2)+(x^2+xy)y'=0$

Soln:

$(3xy+y^2)\,dx+(x^2+xy)\,dy=0$

$M=3xy+y^2 \qquad N=x^2+xy$

Check if $M_y=N_x$

$M_y=\partial_y(3xy+y^2) \qquad N_x=\partial_x(x^2+xy)$

$\quad =3x+2y \qquad\qquad\quad =2x+y$

$M_y \neq N_x$

Multiply both sides of the eqn by $\mu(x)$.

$\mu(3xy+y^2)\,dx+\mu(x^2+xy)\,dy=0$

Assume that $\partial_y(\mu M)=\partial_x(\mu N)$

$\partial_y(\mu M)=\mu\,\partial_y(M)$

$\partial_x(\mu N)=\mu' N+\mu\,\partial_x(N)$

$\mu\,\partial_y(M)=\mu' N+\mu\,\partial_x(N)$

$\mu' N=\mu\,\partial_y(M)-\mu\,\partial_x(N)$

$\quad =\mu(\partial_y M-\partial_x N)$

$\mu'=\dfrac{\mu(\partial_y M-\partial_x N)}{N}$

$\dfrac{\mu'}{\mu}=\dfrac{\partial_y M-\partial_x N}{N}$

Note: For this to be solvable, $\dfrac{\partial_y M-\partial_x N}{N}$ cannot have any y in it.

I.e. $\dfrac{\partial_y M-\partial_x N}{N}$ depends on x only.

$$\frac{\mu'}{\mu} = \frac{(3x+2y)-(2x+y)}{x^2+xy}$$

$$= \frac{x+y}{x(x+y)}$$

$$= \frac{1}{x}$$

$$(\ln(\mu))' = \frac{1}{x}$$

$$\ln(\mu) = \int \frac{1}{x}\, dx$$

$$= \ln|x| + c$$

$$\mu = x$$

Now we have $x(3xy+y^2)\,dx + x(x^2+xy)\,dy = 0$.
Treat it as an exact eqn and solve for F.

$M_1 = x(3xy+y^2)$, $N_1 = x(x^2+xy)$
Let $M_1 = \partial_x F$, $N_1 = \partial_y F$

$$\partial_x F = M_1$$

$$F = \int 3x^2y + xy^2\, dx$$

$$= x^3y + \frac{x^2y^2}{2} + c(y)$$

$$\partial_y F = N_1$$

$$x^3 + x^2y = \partial_y\left(x^3y + \frac{x^2y^2}{2} + c(y)\right)$$

$$= x^3 + x^2y + c'(y)$$

$$c'(y) = 0$$

$$c = \int 0$$

$$= 0 + c_1$$

Let $c_1 = 0$

$$F = x^3y + \frac{x^2y^2}{2} = C$$

E.g. 21 Find $\mu(y)$ and solve
$y + (2xy - e^{-2y})y' = 0$

Soln:

$y\,dx + (2xy - e^{-2y})\,dy = 0$
$\mu(y)y\,dx + \mu(y)(2xy - e^{-2y})\,dy = 0$

$\partial y(\mu M) = \partial x(\mu N)$
$\partial y(\mu M) = \mu' M + \mu\,\partial y(M)$
$\partial x(\mu N) = \mu\,\partial x(N)$
$\mu' M + \mu\,\partial y(M) = \mu\,\partial x(N)$
$\mu' M = \mu\,\partial x(N) - \mu\,\partial y(M)$
$= \mu(\partial x(N) - \partial y(M))$
$\dfrac{\mu'}{\mu} = \dfrac{(\partial x(N) - \partial y(M))}{M}$
$= \dfrac{2y-1}{y}$
$= 2 - \frac{1}{y}$
$(\ln(\mu))' = 2 - \frac{1}{y}$
$\ln(\mu) = \int 2 - \frac{1}{y}\,dy$
$= 2y - \ln|y| + c$

Note: Since μ is a function of y, $\dfrac{\partial x(N) - \partial y(M)}{M}$ should have no terms with x after simplifying.

$\mu = e^{2y - \ln|y| + c}$
$= e^c(e^{2y - \ln|y|})$
$= c'(e^{2y - \ln|y|})$
$= \dfrac{e^{2y}}{y}$

Now, we have
$\left(\dfrac{e^{2y}}{y} \cdot y\right)dx + \left(\dfrac{e^{2y}}{y}\left(2xy - e^{-2y}\right)\right)dy = 0$

$$M_1 = \frac{e^{2y}}{y} \cdot y, \quad N_1 = \frac{e^{2y}}{y}\left(2xy - e^{-2y}\right)$$
$$= e^{2y} \qquad\qquad = 2xe^{2y} - \frac{1}{y}$$

$M_1 = \partial x F$, $N_1 = \partial y F$

$\partial x F = e^{2y}$

$F = \int e^{2y}\, dx$

$= xe^{2y} + c(y)$

$N_1 = \partial y F$

$$2xe^{2y} - \frac{1}{y} = \partial y\left(xe^{2y} + c(y)\right)$$
$$= 2xe^{2y} + c'(y)$$
$$c'(y) = \frac{-1}{y}$$
$$c = \int \frac{-1}{y}\, dy$$
$$= -\ln|y| + c_1$$

let $c_1 = 0$

$$= -\ln|y|$$

$F = xe^{2y} - \ln|y| = c$

Note: If $\dfrac{M_y - N_x}{N}$ depends on x, then

μ is a function of x.

If $\dfrac{N_x - M_y}{M}$ depends on y, then

μ is a function of y.

E.g. 22 Solve $(2x+3)+(2y-2)y'=0$

Soln:

$(2x+3)\,dx+(2y-2)\,dy=0$

$M=2x+3,\ N=2y-2$

Check if $M_y=N_x$

$M_y=\partial_y(M)$
$\quad=\partial_y(2x+3)$
$\quad=0$

$N_x=\partial_x(N)$
$\quad=\partial_x(2y-2)$
$\quad=0$

$M_y=N_x$

Let $M=\partial_x F$ and $N=\partial_y F$

$\partial_x F=2x+3$

$F=\int 2x+3\,dx$
$\quad=x^2+3x+C(y)$

$N=\partial_y F$

$2y-2=\partial_y(x^2+3x+C(y))$
$\quad=C'(y)$

$C=\int 2y-2\,dy$
$\quad=y^2-2y+C_1$

Let $C_1=0$

$F=x^2+3x+y^2-2y=C$

E.g. 23 Solve $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$

Soln:

$(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$

$M = 3x^2 - 2xy + 2 \qquad N = 6y^2 - x^2 + 3$

Check if $M_y = N_x$

$M_y = \partial_y(M)$
$= \partial_y(3x^2 - 2xy + 2)$
$= -2x$

$N_x = \partial_x(N)$
$= \partial_x(6y^2 - x^2 + 3)$
$= -2x$

$M_y = N_x$

Let $M = \partial_x F$ and $N = \partial_y F$.

$\partial_x F = M$
$= 3x^2 - 2xy + 2$

$F = \int 3x^2 - 2xy + 2\,dx$
$= x^3 - x^2y + 2x + C(y)$

$\partial_y F = N$

$6y^2 - x^2 + 3 = \partial_y(x^3 - x^2y + 2x + C(y))$
$= -x^2 + C'(y)$

$C'(y) = 6y^2 + 3$

$C = \int 6y^2 + 3\,dy$
$= 2y^3 + 3y + C_1$

Let $C_1 = 0$

$F = x^3 - x^2y + 2x + 2y^3 + 3y = c$

E.g. 24 Solve $(3x^2y + 2xy + y^3) + (x^2+y^2)y' = 0$

Soln:

$(3x^2y + 2xy + y^3)dx + (x^2+y^2)dy = 0$

$M = 3x^2y + 2xy + y^3 \qquad N = x^2 + y^2$

Check if $M_y = N_x$

$$M_y = \partial_y(M) = \partial_y(3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2$$

$$N_x = \partial_x(N) = \partial_x(x^2+y^2) = 2x$$

$M_y \neq N_x$

$\mu(x)(3x^2y + 2xy + y^3)dx + \mu(x)(x^2+y^2)dy = 0$

Check if $\partial_y(\mu M) = \partial_x(\mu N)$

$\partial_y(\mu M) = \mu(\partial_y(M)) \qquad \partial_x(\mu N) = \mu' N + \mu(\partial_x(N))$

$\mu(\partial_y(M)) = \mu' N + \mu(\partial_x(N))$

$\mu' N = \mu(\partial_y M) - \mu(\partial_x N)$

$$\frac{\mu'}{\mu} = \frac{\partial_y M - \partial_x N}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2+y^2} = 3$$

$(\ln(\mu))' = 3$

$\ln(\mu) = \int 3\, dx = 3x$

$\mu = e^{3x}$

$$e^{3x}(3x^2y + 2xy + y^3)\,dx + e^{3x}(x^2 + y^2)\,dy = 0$$

$M_1 = e^{3x}(3x^2y + 2xy + y^3)$
$N_1 = e^{3x}(x^2 + y^2)$

Let $M_1 = \partial_x F$ and $N_1 = \partial_y F$
$\partial_x F = e^{3x}(3x^2y + 2xy + y^3)$
$F = \int e^{3x}(3x^2y + 2xy + y^3)\,dx$
$= \int 3x^2ye^{3x}\,dx + \int 2xye^{3x}\,dx + \int y^3e^{3x}\,dx$

$\int 3x^2ye^{3x}\,dx = 3y\int x^2e^{3x}\,dx$
Let $u = x^2$, $v = e^{3x}$
$u\int v - \int(u'\int v\,dx)\,dx$
$= x^2\int e^{3x}\,dx - \int((x^2)'\int e^{3x}\,dx)\,dx$
$= \dfrac{x^2e^{3x}}{3} - \int \dfrac{2}{3}xe^{3x}\,dx$
$= \dfrac{x^2e^{3x}}{3} - \dfrac{2}{3}\int xe^{3x}\,dx$
Let $u = x$, $v = e^{3x}$
$u\int v\,dx - \int(u'\int v\,dx)\,dx$
$\dfrac{xe^{3x}}{3} - \int\left(\int e^{3x}\,dx\right)dx$
$\dfrac{xe^{3x}}{3} - \dfrac{e^{3x}}{9}$

$\int 3x^2ye^{3x}\,dx = 3y\left(\dfrac{x^2e^{3x}}{3} - \dfrac{2}{3}\left(\dfrac{xe^{3x}}{3} - \dfrac{e^{3x}}{9}\right)\right)$
$= ye^{3x}\left(x^2 - \dfrac{2x}{3} + \dfrac{2}{9}\right)$
$= ye^{3x}\left(\dfrac{9x^2 - 6x + 2}{9}\right)$

$$\int 2xye^{3x}\,dx = 2y\int xe^{3x}\,dx$$
$$= 2y\left(\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right)$$
$$= 2ye^{3x}\left(\frac{x}{3} - \frac{1}{9}\right)$$
$$= \frac{2ye^{3x}(3x-1)}{9}$$

$$\int y^3e^{3x}\,dx = y^3\int e^{3x}\,dx$$
$$= \frac{y^3e^{3x}}{3}$$

$$ye^{3x}\left[\frac{9x^2-6x+2}{9} + \frac{6x-2}{9} + \frac{3y^2}{9}\right] + c(y)$$

$$ye^{3x}\left[\frac{9x^2+3y^2}{9}\right] + c(y)$$

$$ye^{3x}\left[\frac{3x^2+y^2}{3}\right] + c(y)$$

Hence, $F = ye^{3x}\left[\frac{3x^2+y^2}{3}\right] + c(y)$

$N_1 = \partial y F$

$$e^{3x}(x^2+y^2) = \partial y\left(ye^{3x}\left[\frac{3x^2+y^2}{3}\right] + c(y)\right)$$
$$= \frac{e^{3x}}{3}\left(\partial y\left(3x^2y+y^3\right)\right) + c'(y)$$
$$= \frac{e^{3x}}{3}\left(3x^2+3y^2\right) + c'(y)$$
$$= e^{3x}(x^2+y^2) + c'(y)$$

$$c'(y) = 0 \rightarrow c = \int 0 = 0 \rightarrow F = \frac{ye^{3x}(3x^2+y^2)}{3} = C$$

E.g. 25 Find an integrating factor depending on xy and solve the eqn

$$\left(3x+\frac{6}{y}\right)+\left(\frac{x^2}{y}+\frac{3y}{x}\right)y'=0$$

Soln:

$$\left(3x+\frac{6}{y}\right)dx+\left(\frac{x^2}{y}+\frac{3y}{x}\right)dy=0$$

$$\mu(xy)\left(3x+\frac{6}{y}\right)dx+\mu(xy)\left(\frac{x^2}{y}+\frac{3y}{x}\right)dy=0$$

$$\partial_y(\mu M)=\partial_x(\mu N)$$

$$\partial_y(\mu M)= \quad x\mu' M+\mu(\partial_y M)$$

$$\partial_x(\mu N)= \quad y\mu' N+\mu(\partial_x N)$$

$$x\mu' M+\mu(\partial_y M)=y\mu' N+\mu(\partial_x N)$$

$$x\mu' M-y\mu' N=\mu(\partial_x N)-\mu(\partial_y M)$$

$$\mu'(xM-yN)=\mu(\partial_x N-\partial_y M)$$

$$\frac{\mu'}{\mu}=\frac{\partial_x N-\partial_y M}{xM-yN}$$

$$(\ln(\mu))'= \quad \frac{1}{xy}$$ ← We will let $t=xy$

$$\int(\ln(\mu))'\,dt=\int\frac{1}{t}\,dt$$

$\ln(\mu) = \ln|t| + c$

$\mu = t$

$= xy$

Now we have

$(xy)\left(3x + \frac{6}{y}\right) dx + (xy)\left(\frac{x^2}{y} + \frac{3y}{x}\right) dx$

$M_1 = 3x^2y + 6x \qquad N_1 = x^3 + 3y^2$

let $\partial_x F = M_1$ and $\partial_y F = N_1$

$\partial_x F = 3x^2y + 6x$

$F = \int 3x^2y + 6x \, dx$

$= x^3y + 3x^2 + C(y)$

$N_1 = \partial_y F$

$x^3 + 3y^2 = \partial_y(x^3y + 3x^2 + C(y))$

$= x^3 + C'(y)$

$C'(y) = 3y^2$

$C = \int 3y^2 \, dy$

$= y^3 + C_1$

let $C_1 = 0$

$F = x^3y + 3x^2 + y^3 = c$

E.g. 26 Solve $\cos y\, dx - (x\sin y - y^2)\,dy = 0$

Soln:

$\cos y\, dx + (y^2 - x\sin y)\,dy = 0$

$M = \cos y \qquad N = y^2 - x\sin y$

Check if $M_y = N_x$

$M_y = \partial_y(\cos y) \qquad N_x = \partial_x(y^2 - x\sin y)$
$\quad = -\sin y \qquad\qquad\quad = -\sin y$

$M_y = N_x$

Let $M = \partial_x F$ and $N = \partial_y F$

$\partial_x F = \cos y$
$F = \int \cos y\, dx$
$\;\; = x\cos y + c(y)$

$N = \partial_y F$
$y^2 - x\sin y = \partial_y(x\cos y + c(y))$
$\qquad = -x\sin y + c'(y)$
$c'(y) = y^2$
$c = \int y^2\, dy$
$\;\; = \frac{y^3}{3} + c_1$

Let $c_1 = 0$

$F = x\cos y + \frac{y^3}{3} = c$

E.g. 27 Solve $(3x^2y + 8xy^2)\,dx + (x^3 + 8x^2y + 12y^2)\,dy = 0$

Soln:

$M = 3x^2y + 8xy^2$, $N = x^3 + 8x^2y + 12y^2$

Check if $M_y = N_x$

$$M_y = \partial_y(3x^2y + 8xy^2) = 3x^2 + 16xy \qquad N_x = \partial_x(x^3 + 8x^2y + 12y^2) = 3x^2 + 16xy$$

$M_y = N_x$

Let $\partial_x F = M$ and $\partial_y F = N$

$$\partial_x F = M = 3x^2y + 8xy^2$$

$$F = \int 3x^2y + 8xy^2\,dx = x^3y + 4x^2y^2 + c(y)$$

$$N = \partial_y F$$

$$x^3 + 8x^2y + 12y^2 = \partial_y(x^3y + 4x^2y^2 + c(y)) = x^3 + 8x^2y + c'(y)$$

$$c'(y) = 12y^2$$

$$c = \int 12y^2\,dy = 4y^3 + c_1$$

let $c_1 = 0$

$$F = x^3y + 4x^2y^2 + 4y^3 = c$$

E.g. 28 Solve $(e^x \sin y + e^{-y})dx - (xe^{-y} - e^x \cos y)\,dy = 0$

Soln:

$M = e^x \sin y + e^{-y}$ $\quad N = -xe^{-y} + e^x \cos y$

Check if $M_y = N_x$

$M_y = \partial_y (e^x \sin y + e^{-y})$ $\quad N_x = \partial_x(-xe^{-y} + e^x \cos y)$

$\quad = -e^{-y} \quad e^x \cos y$ $\quad\quad = -e^{-y} + e^x \cos y$

$M_y = N_x$

Let $\partial_x F = M$ and $\partial_y F = N$

$\partial_x F = e^x \sin y + e^{-y}$

$\quad F = \int e^x \sin y + e^{-y}\, dx$

$\quad\quad = e^x \sin y + \quad xe^{-y} + c(y)$

$\partial_y F = N$

$-xe^{-y} + e^x \cos y = \partial_y (e^x \sin y + \quad xe^{-y} + c(y))$

$\quad\quad = e^x \cos y \quad - xe^{-y} + c'(y)$

$c'(y) = 0$

$c = \int 0\, dx$

$= 0 + C_1$

Let $C_1 = 0$

$F = e^x \sin y + xe^{-y} = C$

E.g. 29 Solve $(2x + y\cos x)dx + (2y + \sin x - \sin y)\,dy = 0$

Soln:

$M = 2x + y\cos x \qquad N = 2y + \sin x - \sin y$

Check if $M_y = N_x$

$$M_y = \partial_y M = \partial_y(2x + y\cos x) = \cos x$$

$$N_x = \partial_x N = \partial_x(2y + \sin x - \sin y) = \cos x$$

$M_y = N_x$

Let $\partial_x F = M$ and $\partial_y F = N$

$$\partial_x F = 2x + y\cos x$$

$$F = \int 2x + y\cos x \, dx = x^2 + y\sin x + c(y)$$

$N = \partial_y F$

$$2y + \sin x - \sin y = \partial_y(x^2 + y\sin x + c(y)) = \sin x + c'(y)$$

$$c'(y) = 2y - \sin y$$

$$c = \int 2y - \sin y \, dy = y^2 + \cos y + c_1$$

Let $c_1 = 0$

$$F = x^2 + y\sin x + y^2 + \cos y = C$$

Note: If the question is "Verify that this eqn has an integrating factor that depends on x only.", you must end up with something without y.

Note: If the question is "Check whether this eqn has an integrating factor that depends on x only.", it's trickier than the previous question. In the previous question, you know right away that the integrating factor depends on x only whereas here, you don't.

5. Homogeneous Eqns With Constant Coefficients

- Many second order diff eqns have the form $\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$.

We say that the above eqn is linear if f has the form $f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)y' - q(t)y$.

In this case, we can rewrite the first eqn as

$y'' = g(t) - p(t)y' - q(t)y$

$y'' + p(t)y' + q(t)y = g(t)$

We may also see the eqn in this form:

$P(t)y'' + Q(t)y' + R(t)y = G(t)$

If $P(t) \neq 0$, then we can get $y'' + p(t)y' + q(t)y = g(t)$ from $P(t)y'' + Q(t)y' + R(t)y = G(t)$ by dividing the latter eqn by $P(t)$.

If the eqn is not of the form

$y'' + p(t)y' + q(t)y = g(t)$ or $P(t)y'' + Q(t)y' + R(t)y = G(t)$

then it is non-linear.

- A second order linear diff eqn is homogeneous if $g(t)$ or $G(t)$ or RHS is 0. Otherwise, the eqn is non-homogeneous.

- There are 2 rules used for solving homogeneous eqns with constant coefficients:

Rule 1: Homogeneous eqns always have a soln of this form $y = e^{rt}$ where r is an unknown constant and t is an unknown function.

Rule 2: We can combine solns to get another soln.

- Since $y = e^{rt}$, $y' = re^{rt}$ and $y'' = r^2 e^{rt}$.

Suppose we have $ay'' + by' + cy = 0$.

Now, we have $a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$.

$e^{rt}(ar^2 + br + c) = 0$

Since $e^{rt} \neq 0$, we can divide both sides by it.

Hence, we have

$ar^2 + br + c = 0$ ← Called Characteristic eqn

To find r, we can use the quadratic eqn.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Since $b^2 - 4ac$ could be

a) > 0

b) $= 0$

c) < 0

we have 3 cases to look at.

Case 1: $b^2-4ac>0$

E.g. 30 Solve $y''-y=0$, $y(0)=2$, $y'(0)=-1$

Soln:

$$r^2-1=0$$
$$r^2=1$$
$$r=\pm 1 \rightarrow r_1=1,\ r_2=-1$$

$y_1=e^{r_1 t}$ $\quad y_2=e^{r_2 t}$
$\quad =e^{t}$ $\quad\quad =e^{-t}$

$y=C_1y_1+C_2y_2$
$=C_1e^{t}+C_2e^{-t}$

$y(0)=2$
$2=C_1+C_2$

$y'(0)=-1$
$-1=C_1-C_2$
$C_1=\frac{1}{2}$, $C_2=3/2$
$y=\dfrac{e^{t}}{2}+\dfrac{3e^{-t}}{2}$

Note: When $b^2-4ac>0$, there will be 2 real distinct roots.
I.e. $R_1 \neq R_2$ and $R_1, R_2 \in \mathbb{R}$

E.g. 31 Solve $y'' + 5y' + 6y = 0$, $y(0) = 2$ and $y'(0) = 3$

Soln:

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r_1 = -2$, $r_2 = -3$

$y_1 = e^{r_1 t}$ $\quad$ $y_2 = e^{r_2 t}$

$\quad = e^{-2t}$ $\quad$ $= e^{-3t}$

$y = C_1 y_1 + C_2 y_2$

$\quad = C_1 e^{-2t} + C_2 e^{-3t}$

$y(0) = 2$ $\quad$ $y'(0) = 3$

$2 = C_1 + C_2$ $\quad$ $3 = -2C_1 - 3C_2$

$2 = C_1 + C_2$

$3 = -2C_1 - 3C_2$

$C_1 = 2 - C_2$

$3 = -2(2 - C_2) - 3C_2$

$\quad = -4 + 2C_2 - 3C_2$

$7 = -C_2$

$C_2 = -7$, $C_1 = 9$

$y = 9e^{-2t} - 7e^{-3t}$

Eg. 32 Solve $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = \frac{1}{2}$

Soln:

$4r^2 - 8r + 3 = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{8 \pm \sqrt{64 - 48}}{8}$

$= \frac{8 \pm 4}{8}$

$= \frac{3}{2}$ or $\frac{1}{2}$

$r_1 = 3/2$, $r_2 = 1/2$

$y_1 = e^{r_1 t}$ $\qquad$ $y_2 = e^{r_2 t}$

$= e^{3/2 t}$ $\qquad$ $= e^{t/2}$

$y = c_1 y_1 + c_2 y_2$

$= c_1 e^{3t/2} + c_2 e^{t/2}$

$y(0) = 2$ $\qquad$ $y'(0) = \frac{1}{2}$

$2 = c_1 + c_2$ $\qquad$ $\frac{1}{2} = \frac{3}{2} c_1 + \frac{1}{2} c_2$

$1 = 3c_1 + c_2$

$2 = c_1 + c_2$

$1 = 3c_1 + c_2$

$2c_1 = -1$

$c_1 = -1/2$, $c_2 = 5/2$ $\qquad$ $y = \frac{-e^{3t/2}}{2} + \frac{5e^{t/2}}{2}$

E.g. 33 Solve $y'' + 2y' - 3y = 0$

Soln:

$r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$

$r_1 = -3,\ r_2 = 1$

$y_1 = e^{r_1 t} \qquad y_2 = e^{r_2 t}$

$\quad = e^{-3t} \qquad\quad = e^{t}$

$y = c_1 y_1 + c_2 y_2$

$\quad = c_1 e^{-3t} + c_2 e^{t}$

E.g. 34 Solve $y'' + 3y' + 2y = 0$

Soln:

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r_1 = -2,\ r_2 = -1$

$y_1 = e^{r_1 t} \qquad y_2 = e^{r_2 t}$

$\quad = e^{-2t} \qquad\quad = e^{-t}$

$y = c_1 y_1 + c_2 y_2$

$\quad = c_1 e^{-2t} + c_2 e^{-t}$

E.g. 35 Solve $y'' + 3y' - 10y = 0$

Soln:

$r^2 + 3r - 10 = 0$

$(r+5)(r-2) = 0$

$r_1 = -5,\ r_2 = 2$

$y_1 = e^{r_1 t} \qquad y_2 = e^{r_2 t}$

$\quad = e^{-5t} \qquad\quad = e^{2t}$

$y = c_1 y_1 + c_2 y_2$

$\quad = c_1 e^{-5t} + c_2 e^{2t}$

Case 2: $b^2 - 4ac = 0$

- Here, we only have 1 value for R.

$y_1 = e^{r_1 t}$

$y_2 = te^{r_1 t}$ ← We can use the Wronksian to prove this.

- The Wronksian is also important for seeing if a particular soln is also a general soln. We will look at this first.

Consider the following:

$p(t)y'' + q(t)y' + r(t)y = 0$ and $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

Suppose that y_1 and y_2 are solns to this eqn. We know that $y = C_1 y_1 + C_2 y_2$ is also a soln. What we want to know is if it is also a general soln. In order for it to be a general soln, it must satisfy the initial conditions.

$$y_0 = C_1 y_1(t_0) + C_2 y_2(t_0)$$
$$y'_0 = C_1 y'_1(t_0) + C_2 y'_2(t_0)$$

We can use Cramer's Rule to find C_1 and C_2.

$$C_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \qquad C_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}$$

The denominator is the **Wronksian**.

I.e. $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= y_1 y_2' - y_1' y_2$

Notice that the only thing preventing us from finding C_1 and C_2 is if the denominator (The Wronksian) $=0$. If $W \neq 0$ and y_1 and y_2 are solns, then the 2 solns are called a **fundamental pair/set of solns** and $y = C_1 y_1 + C_2 y_2$ is a general soln.

I.e. Take $y = C_1 y_1 + C_2 y_2$ and the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

$$\left.\begin{array}{l} C_1 y_1(t_0) + C_2 y_2(t_0) = y_0 \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) = y_0' \end{array}\right\}$$ This system has a soln for any RHS iff $W \neq 0$.

- The Wronksian Dichotomy for 2 Solns states that for 2 solns, $W(t) = 0$ for all t or $W(t) \neq 0$ for all t.

- There's a second method to find W.

$W = y_1 y_2' - y_1' y_2$

$W' = (y_1 y_2' - y_1' y_2)'$

$= \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_1' y_2'} - y_1'' y_2$

$= y_1 y_2'' - y_1'' y_2$

Recall that

a) $y_1'' + p(t)y_1' + q(t)y_1 = 0 \longrightarrow y_1'' = -p(t)y_1' - q(t)y_1$

b) $y_2'' + p(t)y_2' + q(t)y_2 = 0 \longrightarrow y_2'' = -p(t)y_2' - q(t)y_2$

So, now we have

$$
\begin{aligned}
w' &= y_1(-p(t)y_2' - q(t)y_2) - y_2(-p(t)y_1' - q(t)y_1) \\
&= -p(t)y_1y_2' - \cancel{q(t)y_1y_2} + p(t)y_1'y_2 + \cancel{q(t)y_1y_2} \\
&= -p(t)y_1y_2' + p(t)y_1'y_2 \\
&= -p(t)\underbrace{(y_1y_2' - y_1'y_2)}_{w} \\
&= -p(t)w
\end{aligned}
$$

$$\frac{dw}{dt} = -p(t)w$$

$$\frac{1}{w}\,dw = -p(t)\,dt \longleftarrow \text{Separable Equation}$$

$$\int \frac{1}{w}\,dw = \int -p(t)\,dt$$

$$\ln|w| + C_1 = -\int p(t)\,dt$$

$$\ln|w| = C_1 - \int p(t)\,dt$$

$$
\begin{aligned}
w &= e^{C_1 - \int p(t)\,dt} \\
&= C' \cdot e^{-\int p(t)\,dt} \longleftarrow \text{Abel's Formula}
\end{aligned}
$$

Note: Abel's formula proves the dichotomy. Either $C' = 0$ and $w = 0$ everywhere or $C' \neq 0$ and $w \neq 0$ everywhere.

E.g. 36 Let $f(t) = e^{2t}$ and $\omega = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$
$= 3e^{4t}$

Solve for $g(t)$.

Soln:

$\omega = fg' - f'g$

$e^{2t}g' - 2e^{2t}g = 3e^{4t}$

$e^{2t}(g' - 2g) = 3e^{4t}$

$g' - 2g = 3e^{2t} \leftarrow$ Linear First Order Diff Eqn

$\mu g' - 2\mu g = 3\mu e^{2t}$

$\cancel{\mu g'} - 2\mu g = (\mu g)'$

$= \mu' g + \cancel{\mu g'}$

$-2\mu \cancel{g} = \mu' \cancel{g}$

$-2 = \dfrac{\mu'}{\mu}$

$= (\ln(\mu))'$

$\int -2\,dt = \int (\ln(\mu))'\,dt$

$-2t + C_1 = \ln(\mu) + C_2$

$-2t + C_1 - C_2 = \ln(\mu)$

$\ln(\mu) = -2t + C$

$\mu = e^{-2t+C}$

$= e^{C} \cdot e^{-2t}$

$= C' \cdot e^{-2t}$

Let $C' = 1$

$\mu = e^{-2t}$

$(\mu g)' = 3\mu e^{2t}$

$(e^{-2t}g)' = 3$

$\int (e^{-2t}g)'\,dt = \int 3\,dt$

$e^{-2t}g + C_1 = 3t + C_2$

$e^{-2t}g = 3t + C_2 - C_1$

$= 3t + C$

$g = 3te^{2t} + \underline{C}e^{2t}$

Note: We keep the C

E.g. 37 Let $x^2y'' - x(x+2)y' + (x+2)y = 0$

a) Verify $y_1 = x$ and $y_2 = xe^x$

Soln:

Simply plug y_1 and y_2 into the LHS and see if LHS = RHS.

y_1:

$LHS = x^2(x)'' - x(x+2)(x)' + (x+2)(x)$

$= -x(x+2) + x(x+2)$

$= 0$

$RHS = 0$

$\therefore$ LHS = RHS for y_1

y_2:

$LHS = x^2(xe^x)'' - x(x+2)(xe^x)' + (x+2)(xe^x)$

$= x^2(x''e^x + 2x'e^{x'} + xe^{x''}) - x(x+2)(x'e^x + xe^{x'})$
$+ (x+2)(xe^x)$

$= x^2(2e^x + xe^x) - x(x+2)(e^x + xe^x) + (x+2)(xe^x)$

$= \cancel{2x^2e^x} + \cancel{x^3e^x} - \cancel{x^2e^x} - \cancel{x^3e^x} - \cancel{2xe^x} - \cancel{2x^2e^x} + \cancel{x^2e^x} + \cancel{2xe^x}$

$= 0$

$RHS = 0$

$\therefore$ LHS = RHS for y_2

$\therefore$ y_1 and y_2 are valid solns.

b) Do y_1 and y_2 make a fundamental pair of solns?

Soln:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= x(xe^x)' - (xe^x)(x)'$$
$$= x(e^x + xe^x) - xe^x$$
$$= xe^x + x^2e^x - xe^x$$
$$= x^2e^x$$

$W \neq 0$ IFF $x \neq 0$

Note: We can't use Abel's Formula here bc all it says is that $W=0$ Iff $C'=0$, which doesn't give us enough information.

E.g. 38 Suppose we have $y_1 = e^{-t}$ and $y_2 = 2e^{-t}$. Prove that they are not a fundamental pair of soln.

Soln:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_1' y_2$$
$$= e^{-t}(2e^{-t})' - (e^{-t})'(2e^{-t})$$
$$= (e^{-t})(-2e^{-t}) - (-e^{-t})(2e^{-t})$$
$$= 0$$

E.g. 39 Let $y_1 = t$ and $y_2 = \sin t$. Are y_1 and y_2 a fundamental pair of solns?

Soln:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= y_1 y_2' - y_1' y_2$$
$$= t\cos t - \sin t$$

If $t=0$, $W=0$.

Furthermore, $\left.\begin{array}{l} W(\frac{\pi}{2}) = -1 < 0 \\ W(2\pi) = 2\pi > 0 \end{array}\right\}$ Contradiction

Abel's Formula either gives all positives or all negatives. Hence, y_1 and y_2 cannot be a fundamental pair of solns.

- Now, we will see how the wronksian can help us find y_2 when we have repeated roots.

E.g. 40 Solve $y'' + 2y' + y = 0$

Soln:

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$

$y = e^{r_1 t}$

$= e^{-t}$

We can use the Wronksian to find y_2.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= y_1 y_2' - y_1' y_2$$
$$= (e^{-t}) y_2' + (e^{-t}) y_2$$

$$W = e^{-\int p(t)\, dt}$$
$$= e^{-\int 2\, dt}$$
$$= e^{-2t}$$

$$(e^{-t}) y_2' + (e^{-t}) y_2 = e^{-2t}$$
$$(e^{-t})(y_2' + y_2) = e^{-2t}$$
$y_2' + y_2 = e^{-t}$ ← Linear First Order Diff Eqn
$$\mu y_2' + \mu y_2 = \mu e^{-t}$$
$$\cancel{\mu y_2'} + \mu y_2 = (\mu y_2)'$$
$$= \mu' y_2 + \cancel{\mu y_2'}$$
$$\mu \cancel{y_2} = \mu' \cancel{y_2}$$
$$1 = \frac{\mu'}{\mu}$$
$$= (\ln(\mu))'$$
$$\int 1\, dt = \int (\ln(\mu))'\, dt$$
$$t + c_1 = \ln(\mu) + c_2$$
$$t + c = \ln(\mu)$$
$$\mu = e^t$$

$$(\mu y_2)' = \mu e^{-t}$$
$$= 1$$
$$\int (\mu y_2)'\, dt = \int 1\, dt$$
$$\mu y_2 + c_1 = t + c_2$$
$$\mu y_2 = t + c$$
$$y_2 = \frac{t + c}{e^t}$$
let $c = 1 \rightarrow y_2 = te^{-t}$

Note: If we have $y'' + by' + cy = 0$ and $r_1 = r_2$, then $y_1 = e^{r_1 t}$ and $y_2 = te^{r_1 t}$. This is called the Repeated Roots Rule.

E.g. 41 Solve $y'' - 4y' + 4y = 0$

Soln:

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$

$y_1 = e^{2t}$, $y_2 = te^{2t} \rightarrow y = C_1 e^{2t} + C_2 t e^{2t}$

E.g. 42 Solve $y'' + 14y' + 49y = 0$

Soln:

$r^2 + 14r + 49 = 0$

$(r+7)^2 = 0$

$r = -7$

$y_1 = e^{-7t}$, $y_2 = te^{-7t} \rightarrow y = C_1 e^{-7t} + C_2 t e^{-7t}$

Case 3: $b^2 - 4ac < 0$

- Here, we have complex roots.
- z is a complex number if it can be written in the form: $z = a + ib$, where a and b are real numbers and $i = \sqrt{-1} \Leftrightarrow i^2 = -1$.

a is the real part.

b is the imaginary part. (Does not include i).

i is the imaginary unit.

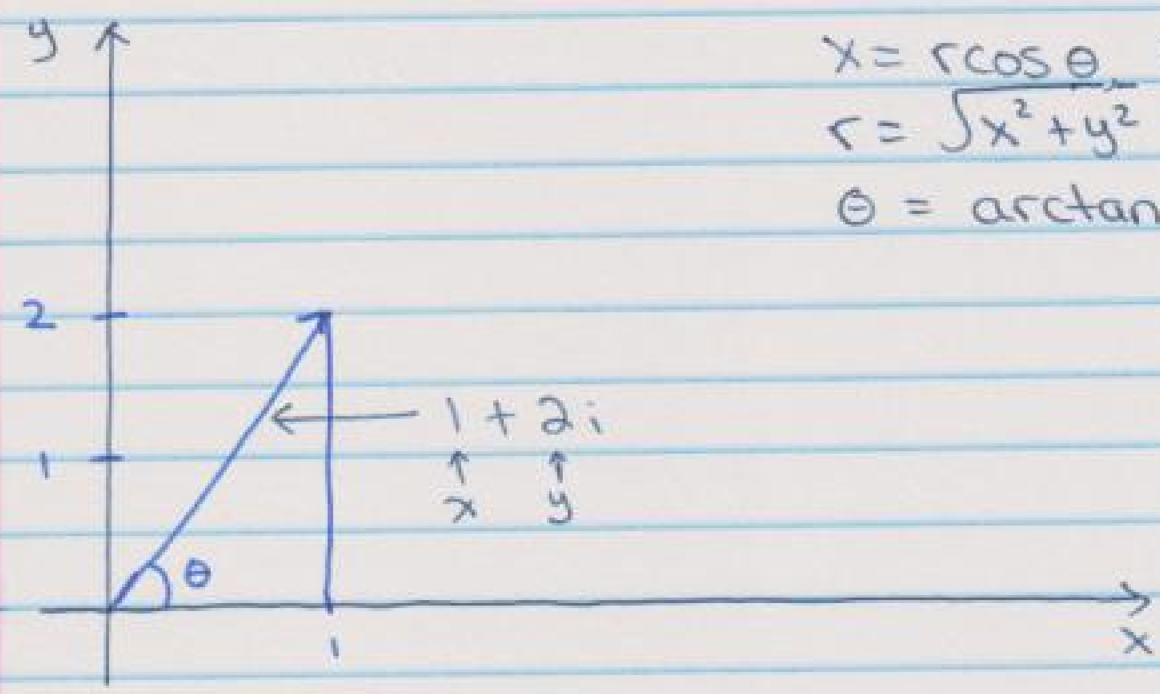
- $(a+ib)+(c+id)=(a+c)+i(b+d)$

E.g. 41 $(1+2i)+(3+4i)=(1+3)+(2+4)i$
$$=4+6i$$

- $(a+ib)\times(c+id) = ac+iad+ibc+i^2bd$
$$= ac+(ad+bc)i-bd$$
$$=(ac-bd)+(ad+bc)i$$

E.g. 42 $(1+2i)\times(3+4i)=3+4i+6i+8i^2$
$$=3-8+10i$$
$$=-5+10i$$

-



$x=r\cos\theta$, $y=r\sin\theta$

$r=\sqrt{x^2+y^2}$

$\theta=\arctan\left(\frac{y}{x}\right)$

x is the real part. y is the imaginary part.

$$z = x+iy$$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta) \leftarrow \text{Polar Form of Complex Numbers}$$

- If $b^2 - 4ac < 0$, then $R_1 = \lambda + iu$ and $R_2 = \lambda - iu$.

$$y = e^{r_1 t}$$
$$= e^{t(\lambda + iu)}$$
$$= e^{\lambda t} \cdot e^{iut}$$

Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

If θ is replaced with $-\theta$, we get

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$
$$= \cos(\theta) - i\sin(\theta)$$

If θ is replaced with $\mu\theta$, we get

$$e^{i\mu\theta} = \cos(\mu\theta) + i\sin(\mu\theta)$$

Hence, $e^{iut} = \cos(ut) + i\sin(ut)$

$$y = e^{\lambda t}(\cos(ut) + i\sin(ut))$$
$$= \underbrace{e^{\lambda t}\cos(ut)}_{y_1} + i\underbrace{e^{\lambda t}\sin(ut)}_{y_2}$$

I.e. $y_1 = e^{\lambda t}\cos(ut)$, $y_2 = e^{\lambda t}\sin(ut)$

E.g. 43 Solve $y'' + y' + 9.25y = 0$

Soln:

$$r^2 + r + \frac{37}{4} = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1^2 - 37}}{2}$$
$$= \frac{-1 \pm 6i}{2}$$
$$= \frac{-1}{2} \pm 3i$$

$r_1 = \frac{-1}{2} + 3i$

$\lambda = \frac{-1}{2}, \quad u = 3$

$y_1 = e^{\lambda t}\cos(ut)$ $\qquad$ $y_2 = e^{\lambda t}\sin(ut)$

$\quad = e^{-t/2}\cos(3t)$ $\qquad$ $= e^{-t/2}\sin(3t)$

$y = C_1 e^{-t/2}\cos(3t) + C_2 e^{-t/2}\sin(3t)$

E.g. 44 Solve $16y'' - 8y' + 145y = 0$

Soln:

$16r^2 - 8r + 145 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{8 \pm 96i}{32}$$

$$= \frac{1}{4} \pm 3i$$

$\lambda = \frac{1}{4}, \quad u = 3$

$y_1 = e^{\lambda t}\cos(ut)$ $\qquad$ $y_2 = e^{\lambda t}\sin(ut)$

$\quad = e^{\frac{t}{4}}\cos(3t)$ $\qquad$ $= e^{\frac{t}{4}}\sin(3t)$

$y = C_1 e^{\frac{t}{4}}\cos(3t) + C_2 e^{\frac{t}{4}}\sin(3t)$

E.g. 45 Solve $y'' + 9y = 0$

Soln:

$r^2 + 9 = 0$

$r^2 = -9$

$r = \pm 3i$

$\lambda = 0, \ u = 3$

$y_1 = e^{\lambda t}\cos(ut)$ $\quad$ $y_2 = e^{\lambda t}\sin(ut)$

$= \cos(3t)$ $\quad$ $= \sin(3t)$

$y = c_1\cos(3t) + c_2\sin(3t)$

E.g. 46 Solve $y'' - 2y' + 2y = 0$

Soln:

$r^2 - 2r + 2 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

$\lambda = 1, \ u = 1$

$y_1 = e^{\lambda t}\cos(ut)$ $\quad$ $y_2 = e^{\lambda t}\sin(ut)$

$= e^t\cos(t)$ $\quad$ $= e^t\sin(t)$

$y = c_1 e^t\cos(t) + c_2 e^t\sin(t)$

E.g. 47 Solve $y'' - 2y' + 6y = 0$

Soln:

$r^2 - 2r + 6 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2 \pm \sqrt{4 - 24}}{2}$$
$$= \frac{2 \pm 2\sqrt{5}}{2}$$
$$= 1 \pm \sqrt{5}$$

$\lambda = 1, \ u = \sqrt{5}$

$y_1 = e^{\lambda t}\cos(ut)$ $\quad$ $y_2 = e^{\lambda t}\sin(ut)$

$\quad = e^{t}\cos(\sqrt{5}t)$ $\quad$ $= e^{t}\sin(\sqrt{5}t)$

$y = c_1 e^{t}\cos(\sqrt{5}t) + c_2 e^{t}\sin(\sqrt{5}t)$

Note: We only use $r_1 = \lambda + iu$, because $\lambda - iu$ gives us the same set of solns.

$\lambda - iu \rightarrow e^{\lambda t}\cos(-ut) + e^{\lambda t}\sin(-ut)$

$\cos(-x) = \cos(x)$ because cos is an even function.

$\sin(-x) = -\sin(x)$ because sin is an odd function.

$y_1 = e^{\lambda t}\cos(ut)$, $y_2 = -e^{\lambda t}\sin(ut)$

$y = c_1 e^{\lambda t}\cos(ut) + c_2(-e^{\lambda t})(\sin(ut))$

The negative sign gets absorbed in c_2.

- More Examples:

E.g. 48 Solve $y'' + 2y' - y = 0$

$r^2 + 2r - 1 = 0$

$\Rightarrow$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - (-4)}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2}$$

$r_1 = -1 + \sqrt{2} \quad r_2 = -1 - \sqrt{2}$

$y_1 = e^{r_1 t} \qquad y_2 = e^{r_2 t}$

$\quad = e^{(-1+\sqrt{2})t} \qquad = e^{(-1-\sqrt{2})t}$

$y = C_1 y_1 + C_2 y_2$

$\quad = C_1 e^{(-1+\sqrt{2})t} + C_2 e^{(-1-\sqrt{2})t}$

E.g. 49 Solve $y'' + 2y' = 0$

$r^2 + 2r = 0$

$r(r+2) = 0$

$r_1 = 0, \; r_2 = -2$

$y_1 = e^{0t}, \; y_2 = e^{-2t}$

$y = C_1 y_1 + C_2 y_2$

$\quad = C_1 + C_2 e^{-2t}$

E.g. 50 Solve $y'' + 2y' - 3y = 0$

$r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$

$r_1 = -3, \quad r_2 = 1$

$y_1 = e^{-3t}, \quad y_2 = e^{t}$

$y = c_1 e^{-3t} + c_2 e^{t}$

E.g. 51 Solve $y'' - 2ky' - 2y = 0$

$r^2 - 2kr - 2 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2k \pm \sqrt{4k^2 + 8}}{2}$$

$$= \frac{2k \pm \sqrt{4(k^2+2)}}{2}$$

$$= \frac{2k \pm 2\sqrt{k^2+2}}{2}$$

$$= k \pm \sqrt{k^2+2}$$

$r_1 = k + \sqrt{k^2+2} \quad r_2 = k - \sqrt{k^2+2}$

$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}$

$$y = c_1 e^{k + \sqrt{k^2+2}\, t} + c_2 e^{k - \sqrt{k^2+2}\, t}$$

E.g. 52 Solve $y'' - 3y' - y = 0$

$r^2 - 3r - 1 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{9 - (-4)}}{2}$$

$$= \frac{3 \pm \sqrt{13}}{2}$$

$\rightarrow y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t}$

$$y = c_1 e^{\frac{3+\sqrt{13}}{2} t} + c_2 e^{\frac{3-\sqrt{13}}{2} t}$$

E.g. 53 Find the fundamental set of solns specified by the initial conditions:
$y(1)=0,\ y'(1)=1$ AND $y(1)=1,\ y'(1)=0$

$$y''+2y'-3y=0$$

$$r^2+2r-3=0$$
$$(r+3)(r-1)=0$$
$$r_1=-3,\ r_2=1$$

$$y=C_1e^{-3t}+C_2e^{t}$$

$y(1)=0,\ y'(1)=1$
$$0=C_1e^{-3}+C_2e$$
$$1=-3C_1e^{-3}+C_2e$$
$$-1=4C_1e^{-3}$$
$$C_1=\frac{-e^{3}}{4}$$
$$C_2e=-C_1e^{-3}$$
$$=-\left(\frac{-e^{3}}{4}\right)e^{-3}$$
$$C_2=\frac{1}{4e}$$

$$y_1=\frac{-e^{3}}{4}e^{-3t}+\frac{e^{t}}{4e}$$
$$=-\frac{1}{4}e^{-3(t-1)}+\frac{1}{4}e^{t-1}$$

$y(1)=1,\ y'(1)=0$

$1 = C_1e^{-3} + C_2e$

$0 = -3C_1e^{-3} + C_2e$

$1 = 4C_1e^{-3}$

$C_1 = \frac{e^3}{4}$

$C_2e = 3C_1e^{-3}$

$= 3\left(\frac{e^3}{4}\right)e^{-3}$

$= \frac{3}{4}$

$C_2 = \frac{3}{4e}$

$y_2 = C_1e^{-3t} + C_2e^t$

$= \frac{e^3}{4}e^{-3t} + \frac{3}{4e}e^t$

$= \frac{1}{4}e^{-3(t-1)} + \frac{3}{4}e^{t-1}$

E.g. 54 Let $f(t) = t$ and $W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = t^2e^t$. Find $g(t)$

$W = fg' - f'g$

$tg' - g = t^2e^t$

$g' - \frac{g}{t} = te^t \leftarrow$ First order Linear Diff eqn

$\mu g' - \frac{\mu g}{t} = \mu te^t$

$\mu g' - \frac{\mu g}{t} = (\mu g)'$

$$\cancel{\mu g'} - \frac{\mu g}{t} = \mu' g + \cancel{\mu g'}$$

$$\frac{-1}{t} = \frac{\mu'}{\mu}$$

$$= (\ln(\mu))'$$

$$\int -\frac{1}{t}\, dt = \int (\ln(\mu))'\, dt$$

$$-\ln|t| + C_1 = \ln(\mu) + C_2$$
$$-\ln|t| + C = \ln(\mu)$$

$$\mu = e^{-\ln|t| + C}$$
$$= e^C (e^{-\ln|t|})$$
$$= C'(t^{-1})$$

let $C' = 1$

$$\mu = \frac{1}{t}$$

$$(\mu g)' = \mu t e^t$$
$$(t^{-1} g)' = e^t$$
$$\int (t^{-1} g)'\, dt = \int e^t\, dt$$
$$t^{-1} g + C_1 = e^t + C_2$$
$$t^{-1} g = e^t + C$$
$$g = te^t + \underline{Ct}$$

Remember to keep the C.

6. Reduction of order:

- Now, we will focus on homogeneous eqns with non-constant coefficients.
- Rule 1: y_1 is given.

I.e. In the question, you'll see $y_1 = \frac{1}{t}$ or $y_1 = t$, etc

- Rule 2: $y_2 = v y_1$ where v is an unknown function.

E.g. 55 Solve $2t^2y'' + 3ty' - y = 0$, $y_1 = \frac{1}{t}$

$y_2 = vy_1$ ⟵ D' Alembert

$2t^2 y_2'' + 3ty_2' - y_2 = 0$

$2t^2(vy_1)'' + 3t(vy_1)' - vy_1 = 0$

$2t^2(v''y_1 + 2v'y_1' + vy_1'') + 3t(v'y_1 + vy_1') - vy_1 = 0$

$2t^2v''y_1 + 4t^2v'y_1' + 2t^2vy_1'' + 3tv'y_1 + 3tvy_1' - vy_1 = 0$

Collect all the terms with v.

$v\underbrace{(2t^2y_1'' + 3ty_1' - y_1)}$

Since y_1 is a soln to the original eqn, this equals 0.

Note: If you did your calculations correctly, all the terms with v will be gone.
"V must go"

We're left with

$2t^2v''y_1 + 4t^2v'y_1' + 3tv'y_1 = 0$

Recall $y_1 = t^{-1}$

$2tv'' - 4v' + 3v' = 0$

$2tv'' - v' = 0$

Let $w = v'$, $w' = v''$

$2tw' - w = 0$

$$2t \frac{dw}{dt} = w$$

$$2t \, dw = w \, dt$$

$$\frac{1}{w} dw = \frac{1}{2t} dt$$

$$\int \frac{1}{w} dw = \int \frac{1}{2t} dt$$

$$\ln|w| + c_1 = \frac{\ln|t|}{2} + c_2$$

$$\ln|w| = \frac{\ln|t|}{2} + c_2 - c_1$$

$$= \frac{\ln|t|}{2} + c$$

$$w = e^{\frac{\ln|t|}{2} + c}$$

$$= e^c \left(e^{\frac{\ln|t|}{2}}\right)$$

$$= c' \cdot t^{1/2}$$

Let $c' = 1$

$$w = t^{1/2}$$

Recall $v' = w \rightarrow v = \int w \, dt$

$$v = \int t^{1/2} \, dt$$

$$= \frac{2t^{3/2}}{3} + c$$

Let $c = 0$

$$v = \frac{2t^{3/2}}{3}$$

$$y_2 = v y_1$$

$$= \frac{2}{3} t^{3/2} \cdot t^{-1}$$

$$= \frac{2t^{1/2}}{3}$$

Note: Using D'Alembert always guarantees that y_1 and y_2 are a fundamental pair of solns.

Note: If the question also asks to verify that y_1 and y_2 are a fundamental pair of soln, show that $W[y_1, y_2] \neq 0$

E.g. 56 Given a soln $y_1(x)$, use D'Alembert to find a second independent soln to
$(2x^2+1)y'' - 4xy' + 4y = 0$, $y_1 = x$

$y_2 = vy_1$
$(2x^2+1)(y_2)'' - 4x(y_2)' + 4y_2 = 0$
$(2x^2+1)(vy_1)'' - 4x(vy_1)' + 4vy_1 = 0$
$(2x^2+1)(v''y_1 + 2v'y_1' + vy_1'') - 4x(v'y_1 + vy_1')$
$+ 4vy_1 = 0$
$(2x^2+1)v''y_1 + (2x^2+1)(2v'y_1') + \cancel{(2x^2+1)vy_1''}$
$- 4x(v'y_1) - \cancel{4x(vy_1')} + \cancel{4vy_1} = 0$

Collect all terms with v.
$v[(2x^2+1)y_1'' - 4xy_1' + 4y_1] = 0$

$(2x^2+1)v''y_1 + (2x^2+1)(2v'y_1') - 4xv'y_1 = 0$
Sub x in for y_1
$(2x^2+1)v''x + (2x^2+1)(2v') - 4x^2v' = 0$
$2x^3v'' + v''x + \cancel{4x^2v'} + 2v' - \cancel{4x^2v'} = 0$
$2x^3v'' + v''x + 2v' = 0$
Let $w = v'$, $w' = v''$
$(2x^3 + x)w' + 2w = 0$
$(2x^3+x)w' = -2w$

$$(2x^3+x)\frac{dw}{dx} = -2w$$

$$(2x^3+x)\,dw = -2w\,.\,dx$$

$$\frac{1}{w}\,dw = \frac{-2}{2x^3+x}\,dx$$

$$\int \frac{1}{w}\,dw = \int \frac{-2}{2x^3+x}\,dx$$

$$\ln|w| + C_1 = \int \frac{-2}{2x^3+x}\,dx$$

$$\ln|w| = \int \frac{-2}{2x^3+x}\,dx + C_1$$

$$w = e^{\int \frac{-2}{2x^3+x}\,dx + C_1}$$

$$= e^{c} \cdot e^{\int \frac{-2}{2x^3+x}\,dx}$$

Let $e^{c} = 1$

$$w = e^{\int \frac{-2}{2x^3+x}\,dx}$$

Recall that $v' = w \longrightarrow v = \int w\,dx$

$$v = \int e^{\int \frac{-2}{2x^3+x}\,dx}\,dx$$

$$y_2 = v \cdot y_1$$

Eg. 57 Given $2x^2y'' + 3xy' - y = 0$ and $y_1 = x^{1/2}$, find y_2

$y_2 = vy_1$

$2x^2 y_2'' + 3x y_2' - y_2 = 0$

$2x^2 (vy_1)'' + 3x (vy_1)' - vy_1 = 0$

$2x^2 (v''y_1 + 2v'y_1' + vy_1'') + 3x(v'y_1 + vy_1') - vy_1 = 0$

$2x^2v''y_1 + 4x^2v'y_1' + \cancel{2x^2vy_1''} + 3xv'y_1 + \cancel{3xvy_1'} - \cancel{vy_1} = 0$

Collect all the terms with v.

$v(2x^2y_1'' + 3xy_1' - y_1) = 0$

"v must go"

We're left with

$2x^2v''y_1 + 4x^2v'y_1' + 3xv'y_1 = 0$

Let $y_1 = x^{1/2}$

$2x^{5/2}v'' + 2x^{3/2}v' + 3x^{3/2}v' = 0$

$2x^{5/2}v'' + 5x^{3/2}v' = 0$

$2xv'' + 5v' = 0$

Let $w = v'$, $w' = v''$

$2xw' + 5w = 0$

$2x \frac{dw}{dx} = -5w$

$2x\, dw = -5w\, dx$

$\frac{1}{w} dw = \frac{-5}{2x} dx$

$\int \frac{1}{w} dw = \int \frac{-5}{2x} dx$

$\ln|w| + C_1 = \frac{-5}{2} \ln|x| + C_2$

$\ln|w| = \frac{-5}{2} \ln|x| + C$

$w = e^{-\frac{5}{2}\ln|x| + c}$

$= e^{c} \cdot e^{-\frac{5}{2}\ln|x|}$

$= c' \quad (x^{-5/2})$

let $c' = 1$

$w = x^{-5/2}$

$v' = w \rightarrow v = \int w \, dx$

$= \int x^{-5/2} \, dx$

$= \frac{-2}{3} x^{-3/2} + c$

let $c = 0$

$y_2 = v y_1$

$= \left(\frac{-2}{3} x^{-3/2}\right) x^{1/2}$

$= \frac{-2}{3x}$

E.g. 58 Given $y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0$ and $y_1 = x$, use D'Alembert to find y_2.

$y_2 = v y_1$

$(v y_1)'' - \frac{2}{x}(v y_1)' + \frac{2}{x^2}(v y_1) = 0$

$v'' y_1 + 2 v' y_1' + v y_1'' - \frac{2}{x}(v' y_1 + v y_1') + \frac{2}{x^2}(v y_1) = 0$

$v'' y_1 + 2 v' y_1' + \cancel{v y_1''} - \frac{2}{x} v' y_1 - \cancel{\frac{2}{x} v y_1'} + \cancel{\frac{2}{x^2} v y_1} = 0$

Collect all terms with v: $v(y_1'' - \frac{2}{x} y_1' + \frac{2}{x^2} y_1) = 0$

$v'' y_1 + 2 v' y_1' - \frac{2}{x} v' y_1 = 0$

let $y_1 = x$

$v'' x + 2v' - 2v' = 0$

$v'' x = 0$

$v'' = 0$

let $w = v'$, $w' = v''$

$w' = 0$

$w = \int 0 \, dx \rightarrow w = c \rightarrow v = \int w \, dx \rightarrow v = \int c \, dx \rightarrow v = x \rightarrow y_2 = v y_1 = x^2$

7. Euler's Equation:

- Consider $t^2 y'' + \alpha t y' + \beta y = 0$ where α and β are known constants.

We let $y = t^r$ where r is an unknown constant. $y' = r t^{r-1}$, $y'' = r(r-1)t^{r-2}$

We now have $t^2 (r)(r-1) t^{r-2} + \alpha t r t^{r-1} + \beta t^r = 0$

$$t^r (r)(r-1) + t^r (\alpha r) + t^r \beta = 0$$
$$t^r \left[r^2 - r + \alpha r + \beta \right] = 0$$
$$r^2 + (\alpha - 1) r + \beta = 0$$

↑ This is called the characteristic eqn for Euler's eqn / Indical eqn

- We can use the quadratic eqn to solve for r. Since $b^2 - 4ac$ has 3 possibilities, we have to deal with each case.

Case 1 $b^2 - 4ac > 0$:

- Here, $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$

E.g. 58 Solve $t^2 y'' + t y' - 2y = 0$

$\alpha = 1$, $\beta = -2$

$r^2 + (\alpha - 1) r + \beta = 0$

$r^2 + (1-1) r - 2 = 0$

$r^2 = 2$

$r = \pm \sqrt{2} \rightarrow r_1 = \sqrt{2}$, $r_2 = -\sqrt{2}$

$y_1 = t^{\sqrt{2}}$, $y_2 = t^{-\sqrt{2}}$

$y = c_1 t_1 + c_2 t_2$

$\quad = c_1 t^{\sqrt{2}} + c_2 t^{-\sqrt{2}}$

Note: If $R_1 \neq R_2$ and $R_1, R_2 \in \mathbb{R}$, then $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$ always make a fundamental pair of solns.

Case 2 $b^2 - 4ac = 0$:
- Here $R_1 = R_2$ (Repeated Roots)
- Here, $y_1 = t^{r_1}$ and $\underline{y_2 = \ln(t) \cdot t^{r_2}}$

Note: We can use D'Alembert to prove this.

E.g. 59 Solve $t^2y'' - 5ty' + 9y = 0$

$\alpha = -5,\ \beta = 9$
$r^2 + (\alpha - 1)r + \beta = 0$
$r^2 + (-5-1)r + 9 = 0$
$r^2 - 6r + 9 = 0$
$(r-3)^2 = 0$
$r = 3$
$y_1 = t^3,\ y_2 = \ln(t) \cdot t^3$
$y = C_1 t^3 + C_2 \ln(t) \cdot t^3$

Case 3 $b^2 - 4ac < 0$:
- Here we have complex roots.

E.g. 60 Solve $t^2y'' + 3ty' + 2y = 0$

$\alpha = 3,\ \beta = 2$
$r^2 + (\alpha - 1)r + \beta = 0$
$r^2 + (3-1)r + 2 = 0$
$r^2 + 2r + 2 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$r_1 = -1 + i$

$y = t^{r_1}$

$= t^{-1+i}$

$= t^{-1} \cdot t^{i}$

$t = e^{\ln t}$

$t^i = e^{i \ln t}$

Hence, $\dfrac{t^i}{t} = \dfrac{e^{i\ln|t|}}{t}$ ← Euler's Formula

$= \dfrac{\cos(\ln|t|) + i\sin(\ln|t|)}{t}$

$y_1 = \dfrac{\cos(\ln(t))}{t}, \quad y_2 = \dfrac{\sin(\ln(t))}{t}$

$y = c_1 \dfrac{\cos(\ln t)}{t} + \dfrac{c_2 \sin(\ln t)}{t}$

8. Non-Linear Homogeneous Eqns

- Has the form $y'' + p(t)y' + q(t)y = g(t)$
- Rule: If y_1 and y_2 are solns to the non-homogeneous eqn, then $y_2 - y_1$ solves the homogeneous eqn.
- Rule: The general soln to a non-homogeneous soln = General soln of the homogeneous soln + particular soln of the non-homogeneous eqn.

- To find the particular soln of the non-homogeneous eqn, we will use the method of undetermined coefficients.

- Undetermined Coefficients
  - Note: This method only works for some functions.

E.g. 61 Find a particular soln to
$y'' - 3y' - 4y = 3e^{2t}$

Let $y = Ae^{2t}$
$(Ae^{2t})'' - 3(Ae^{2t})' - 4Ae^{2t} = 3e^{2t}$
$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$
$-6Ae^{2t} = 3e^{2t}$
$A = -1/2$

$-\frac{1}{2}e^{2t}$ is the particular soln.

To find the general soln,
$y'' - 3y' - 4y = 0$
$r^2 - 3r - 4 = 0$
$(r-4)(r+1) = 0$
$r_1 = 4, r_2 = -1$ ← Important that $R_1, R_2 \neq 2$
$C_1e^{4t} + C_2e^{-t}$ is the general soln

$$y = \underbrace{\frac{-e^{2t}}{2}}_{\text{Particular soln}} + \underbrace{C_1e^{4t} + C_2e^{-t}}_{\text{General soln of homogeneous eqn}}$$

E.g. 62 Find a particular soln to
$y'' - 4y' - 12y = 3e^{5t}$

Let $y = Ae^{5t}$
$(Ae^{5t})'' - 4(Ae^{5t})' - 12Ae^{5t} = 3e^{5t}$
$25Ae^{5t} - 20Ae^{5t} - 12Ae^{5t} = 3e^{5t}$
$-7A = 3$
$A = \frac{-3}{7}$

$\frac{-3e^{5t}}{7}$ is the particular soln

To find the general soln,
$y'' - 4y' - 12y = 0$
$r^2 - 4r - 12 = 0$
$(r-6)(r+2) = 0$
$r_1 = 6,\ r_2 = -2$
$C_1e^{6t} + C_2e^{-2t}$ is the general soln

$$y = \frac{-3e^{5t}}{7} + C_1e^{6t} + C_2e^{-2t}$$

E.g. 63 Find a particular soln to
$y'' - 3y' - 4y = 2\sin(t)$

Let $y = A\cos t + B\sin t$
$(A\cos t + B\sin t)'' - 3(A\cos t + B\sin t)' -$
$4(A\cos t + B\sin t) = 2\sin t$

$-A\cos t - B\sin t + 3A\sin t - 3B\cos t -$
$4A\cos t - 4B\sin t = 2\sin t$

Collect all the terms with sint and cost, individually.

$$\left.\begin{aligned} -A - 3B - 4A &= 0 \\ -B + 3A - 4B &= 2 \end{aligned}\right\} \rightarrow \begin{aligned} -3B - 5A &= 0 \\ 3A - 5B &= 2 \end{aligned}$$

$A = \frac{3}{17}$, $B = \frac{-5}{17}$

$\frac{3}{17}\cos t - \frac{5}{17}\sin t$ is the particular soln

E.g. 64 Find a particular soln to $y'' - 4y' - 12y = \sin(2t)$

Let $y = A\cos(2t) + B\sin(2t)$

$(A\cos(2t))'' + (B\sin(2t))'' - 4(A\cos(2t) + B\sin(2t))' - 12(A\cos(2t) + B\sin(2t)) = \sin(2t)$

$-4A\cos(2t) - 4B\sin(2t) + 8A\sin(2t) - 8B\cos(2t) - 12A\cos(2t) - 12B\sin(2t) = \sin(2t)$

$$\left.\begin{aligned} -4A - 8B - 12A &= 0 \\ -4B + 8A - 12B &= 1 \end{aligned}\right\} \begin{aligned} -8B - 16A &= 0 \\ 8A - 16B &= 1 \end{aligned}$$

$A = \frac{1}{40}$, $B = \frac{-1}{20}$

$\frac{\cos(2t)}{40} - \frac{\sin(2t)}{20}$ is the particular soln

E.g. 65 Find a particular soln to
$y'' - 3y' - 4y = -8e^t \cos(2t)$

Let $y = Ae^t\cos(2t) + Be^t\sin(2t)$

$(Ae^t\cos(2t) + Be^t\sin(2t))''$
$-3(Ae^t\cos(2t) + Be^t\sin(2t))'$
$-4(Ae^t\cos(2t) + Be^t\sin(2t))$
$= -8e^t\cos(2t)$

$A = \frac{10}{13}$, $B = \frac{2}{13}$

E.g. 66 Find a particular soln to
$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos(2t)$

To solve this, simply add up the particular solns for each term on the RHS. This is called superposition of particular solns. In practice, we won't be evaluated on it.

Rule: If y_1 is a particular soln for $y'' + p(t)y' + q(t)y = g_1(t)$ and y_2 is a particular soln for $y'' + p(t)y' + q(t)y = g_2(t)$, then $y_1 + y_2$ is a particular soln to $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$

E.g. 67 Find a particular soln to
$y'' - 3y' - 4y = 2e^{-t}$

Let $y = Ae^{-t}$
$(Ae^{-t})'' - 3(Ae^{-t})' - 4Ae^{-t} = 2e^{-t}$
$Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} = 2e^{-t}$
$0 = 2e^{-t}$

To figure out why we got $0 = 2e^{-t}$,
consider $y'' - 3y' - 4y = 0$
$r^2 - 3r - 4 = 0$
$(r-4)(r+1) = 0$
$r_1 = 4, \; r_2 = -1$
$y = c_1 e^{4t} + \underline{c_2 e^{-t}}$
Same as Ae^{-t}

Since Ae^{-t} solves the homogeneous eqn, it can't also solve the non-homogeneous eqn. This is called a resonance. Here, we let $y = Ate^{-t}$. (Proof follows from D'Alembert)

$(Ate^{-t})'' - 3(Ate^{-t})' - 4Ate^{-t} = 2e^{-t}$
$-2Ae^{-t} + Ate^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$

Collect all the terms with At. They should cancel out.

$Ate^{-t} + 3Ate^{-t} - 4Ate^{-t} = 0$

$-2Ae^{-t} - 3Ae^{-t} = 2e^{-t}$
$A = \frac{-2}{5}$

$\frac{-2}{5} t e^{-t}$ is the particular soln.

E.g. 68 Find a particular soln to
$y'' + 4y' + 4y = e^{-2t}$

Consider $y'' + 4y' + 4y = 0$

$$r^2 + 4r + 4 = 0$$
$$(r+2)^2 = 0$$
$$r = -2$$
$$y = C_1 e^{-2t} + C_2 t e^{-2t}$$

Here, we can't let $y = Ae^{-2t}$ or $y = Ate^{-2t}$. Instead, we'll let $y = At^2 e^{-2t}$. This is called double resonance.

$$(At^2 e^{-2t})'' + 4(At^2 e^{-2t})' + 4At^2 e^{-2t} = e^{-2t}$$
$$2Ae^{-2t} - \cancel{8Ate^{-2t}} + \cancel{4At^2 e^{-2t}} + \cancel{8Ate^{-2t}}$$
$$- \cancel{8At^2 e^{-2t}} + \cancel{4At^2 e^{-2t}} = e^{-2t}$$

$$2Ae^{-2t} = e^{-2t}$$
$$A = \frac{1}{2}$$

$y = \frac{t^2 e^{-2t}}{2}$ is the particular soln

E.g. 69 Find a particular soln to
$y'' + y' + 9.25y = -6e^{-t/2}\cos(3t)$

Consider $y'' + y' + 9.25y = 0$

$$r^2 + r + 9.25 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1-37}}{2}$$
$$= \frac{-1 \pm 6i}{2}$$
$$= -\frac{1}{2} \pm 3i$$

$$y_1 = e^{-t/2}\cos(3t), \quad y_2 = e^{-t/2}\sin(3t)$$
$$y = C_1 e^{-t/2}\cos(3t) + C_2 e^{-t/2}\sin(3t)$$

This is called complex resonance.
Here, we'll let $y = Aty_1 + Bty_2$

$(Aty_1)'' + (Bty_2)'' + (Aty_1 + Bty_2)'$
$+ 9.25(Aty_1 + Bty_2) = -6e^{-t/2}\cos(3t)$

$2Ay_1' + Aty_1'' + 2By_2' + Bty_2'' + Ay_1 +$
$Aty_1' + By_2 + Bty_2' + 9.25Aty_1 +$
$9.25Bty_2 = -6e^{-t/2}\cos(3t)$

Collect all terms with At and Bt.

$$At\underbrace{[y_1'' + y_1' + 9.25y_1]}_{0} = 0$$

$$Bt\underbrace{[y_2'' + y_2' + 9.25y_2]}_{0} = 0$$

$$2Ay_1' + 2By_2' + Ay_1 + By_2 = -6e^{-t/2}\cos(3t)$$

From the previous page, $y_1 = e^{-t/2}\cos(3t)$
$y_2 = e^{-t/2}\sin(3t)$

$$2A(e^{-t/2}\cos(3t))' + 2B(e^{-t/2}\sin(3t))' +$$
$$Ae^{-t/2}\cos(3t) + Be^{-t/2}\sin(3t) = -6e^{-t/2}\cos(3t)$$

$$6B\cos(3t) - 6A\sin(3t) = -6\cos(3t)$$

$A=0$, $B=-1$

$-te^{-t/2}\sin(3t)$ is the particular soln